Analytic Mechanics: Discussion Worksheet 1

Robin Blume-Kohout

September 7, 2000

This worksheet deals with some of the mathematics essential for your understanding of mechanics. We'll work, later on, with techniques of calculus and geometry, but for now we're going to introduce some ideas of linear algebra and some notation.

1 Vectors and matrices

Let the following variables represent matrices (some also represent vectors):

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & +\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \quad E = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad F = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{pmatrix}$$

1.1 Which pairs of the preceding matrices may be added?

1.2 Which pairs of the preceding matrices may be multiplied?

1.3 Of all of the allowed multiplications in the previous question, calculate explicitly both the least complicated (smallest number of scalar multiplications) and most complicated (largest number of scalar multiplications).

2 Tensors and component notation

In this section, all the matrices from the previous section will be considered as tensors. There is a technical question about whether non-square matrices should properly be described as tensors, but this will not be an issue after this point.

2.1 Consider a vector $\vec{a}$. Write (symbolically) the ith component of $\vec{a}$. If $\vec{a}$ is a 3-vector of the form (x, y, z), how would we indicate y in component notation?

2.2 Consider the matrix F from the previous section as a tensor. We will denote the number in the ith row and the jth column as F_{ij}. Can you find a formula that gives F_{ij} for all appropriate i and j?

2.3 If each matrix in the previous section is considered as a tensor, what is the rank of each?

a) b) c) d) e) f)

2.4 If the matrices in the previous section are considered as tensors, which are "symmetric" and which are "antisymmetric?" For which does the question even make sense?

a) b) c)

d) e) f)

2.5 The Levi-Civita tensor ϵ is described as "the totally antisymmetric unit tensor of rank 3." Given this definition, the additional (technically unnecessary) hint that its indices range from 1 to 3, and the (necessary) convention that $\epsilon_{123} = 1$, can you write down all the components of ϵ?

3 Tensor contraction, and thence to matrix multiplication

3.1 **Given the matrices A and B below,calculate $C = A \cdot B$ and $D = B \cdot A$. Now, write a component-notation formula for C_{ij} (feel free to use subscripts like k, l, m, n, a, b, c, etc.) and D_{ij}. Check a few components of C and D to make sure that your formula works (don't forget to sum!) If we let $E = C \cdot A + D \cdot B$, what is E_{ij} in terms of components of A and B?**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & +1 \\ -1 & 2 \end{pmatrix}$$

$C =$ $\qquad\qquad$ $D =$

$C_{ij} =$ $\qquad\qquad$ $D_{ij} =$

$E_{ij} =$

3.2 **P, Q, and R are square 2-tensors (from now on, all tensors are square, cubic, hypercubic, etc.) of the same order (dimensionality, or size). P and Q are symmetric, R is antisymmetric. Classify the following as symmetric, antisymmetric, or neither. Justify your reasoning (i.e., show your work).**

3.2.1 $X : X_{ij} = P_{ik}Q_{kj} + Q_{ik}P_{kj}$

3.2.2 $Y : Y_{ij} = P_{ik}Q_{kj} - Q_{ik}P_{kj}$

3.2.3 $Z : Z_{ij} = Q_{ik}R_{kj} - R_{ik}Q_{kj}$

3.2.4 $T : T_{ij} = P_{ik}Q_{kl}R_{lm}Q_{mn}P_{nj}$

3.3 Let u and v be 1-tensors. Let S and T be 2-tensors. What sort of entity is each of the following?

3.3.1 $u_i S_{ij}$

3.3.2 $T_{ij} u_i$

3.3.3 $u_i v_j$

3.3.4 $u_i v_i$

3.3.5 $T_{ij} S_{ji}$

3.3.6 $u_i T_{ij} S_{jk} v_k$

3.3.7 $u_i T_{ij} v_j$

3.3.8 $u_i T_{jk} S_{lm} v_k$

4 Give several (3 or more) definitions of what a tensor is. Strive for intuition and clarity, not precision – this isn't a graded exercise!